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1995 J. Phys. A: Math. Gen. 28 L317

(http://iopscience.iop.org/0305-4470/28/11/002)

J. Phys. A: Math. Gen. 28 (1995) L317–L322. Printed in the UK

LETTER TO THE EDITOR

Geometry of force-free fields

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Received 5 April 1995

Abstract. A geometric generalization of force-free fields is proposed. Properties of the minimum-energy configuration are deduced and its connection with minimal surfaces is shown for the simplest case.

Forty years ago Lust and Schlute [1] introduced the concept of force-free magnetic fields ($\nabla \times B = cB$) to describe large currents and magnetic fields co-existing in matter with no force bearing on the material. Since then, there has been a gradual and steady growth in the literature on this subject. Force-free fields have been used in many different applications including the study of solar flares [2], superconductors [3] and plasma confinement [4].

There is a growing awareness of the utility of geometric generalizations in theoretical work; one feels that some useful results could be derived from a geometrically inspired treatment of force-free fields. We will see that there is more than an accidental connection between force-free fields and the burgeoning subject of minimal surfaces, a connection that does not seem to have been noted previously.

As is well known, the equilibrium equations describing magnetohydrodynamical fields are nonlinear and seemingly intractable. Any geometric input is a welcome means of simplifying the equations since they are generally independent of the dynamics. Recently topological invariants and topologically-related quantities have been found to be useful in obtaining lower bounds on magnetic-field energies [5, 6]. This letter presents a framework, based on differential forms, with which to formulate the theory of force-free fields. Because of its simplicity it gives another perspective into the theory and should encourage others to consider the insights gained from the study of minimal surfaces within the context of force-free fields.

Magnetic fields are divergence-free vector fields [7]. We will, therefore, begin by considering the divergence free $(n-1)$ forms α in a $(2n-1)$ compact manifold M (with suitable gauge condition if necessary):

$$d^*\alpha = 0 \tag{1}$$

where d is the exterior derivative and $*$ the Hodge star operator mapping p-forms to $(2n-1-p)$-forms [8]. Additionally, we will require that the Lie derivative of α with respect to a divergence-free vector field X be exact:

$$L_X\alpha = d\phi \qquad \operatorname{div} X = 0. \tag{2}$$

Here ϕ is an $(n-2)$-form, L_X denotes the Lie derivative and we assume that the quantities defined are sufficiently continuous.

0305-4470/95/110317+06$19.50

The second condition given above is really a generalization of the frozen-in condition for magnetic fields and vortices in fluids [9]. For instance, the Euler equation for a zero-viscosity incompressible fluid can be written as

$$L_{\partial/\partial t+v}u = \mathrm{d}\left(\tfrac{1}{2}u^2 + \frac{p}{\rho} + \theta\right) \tag{3}$$

where v is the velocity field, $u = v^i\,\mathrm{d}x^i$ is the same quantity written as a 1-form, p is the pressure, ρ is the density and θ is an external conserved body potential. Similarly, an incompressible perfectly conducting fluid obeys the Navier–Stokes equation in the form $L_{\partial/\partial t+v}B = 0$, where B is the magnetic field. This is essentially the Kelvin–Helmholtz theorem for the conservation of vorticity.

The vector fields X_i form a Lie algebra with respect to Lie brackets because

$$L_{[X_i,X_j]}\alpha = (L_{X_i}L_{X_j} - L_{X_j}L_{X_i})\alpha = \mathrm{d}(L_{X_i}\phi_j - L_{X_j}\phi_i) \tag{4}$$

and because $[X_i, X_j]$ is divergence free if X_i and X_j satisfy the identity

$$\mathrm{div}[X_i, X_j] = X_i\,\mathrm{div}\,X_j - X_j\,\mathrm{div}\,X_i. \tag{5}$$

Defining the n-form B as

$$B = \mathrm{d}\alpha \tag{6}$$

we immediately see that $L_X B = 0$ which generalizes the Kelvin–Helmholtz theorem mentioned above.

We must consider the implications raised by consistency. If we require that $[X_i, X_j] = f_{ijk}X_k$ where the fs are constants, then we find from (4) and (1) that

$$L_{X_i}\phi_j - L_{X_j}\phi_i - f_{ijk}\phi_k = 0. \tag{7}$$

This is the consistency condition that must be satisfied by the ϕ_i. The coefficients f_{ijk} are antisymmetric in the first two indices and obey the Jacobi identity

$$f_{ijk}f_{kmn} + f_{jmk}f_{kin} + f_{mik}f_{kjn} = 0. \tag{8}$$

Next we introduce the energy E_B and the Hopf invariant S_{AB} [9]:

$$E_B = \int_M^* B \wedge B \equiv (B, B) \qquad S_{AB} = \int_M \alpha\,\mathrm{d}\alpha. \tag{9}$$

Using (1) and (2) we can verify that

$$L_X S_{AB} = 0 \tag{10}$$

that is, S_{AB} is invariant under deformations generated by incompressible flows. S_{AB} does not depend on the specific choice of α and describes the helicity of the field. If v denotes a Killing field, it is known that $L_v^* = {}^*L_v$ so that

$$L_v E_B = 2\int^* B \wedge L_v B. \tag{11}$$

The Killing field is divergence free so both S_{AB} and E_B are invariant under flows arising from v (i.e. from isometries).

The field energy attains its critical value when

$$\mathrm{d}\alpha = \lambda^*\alpha \tag{12}$$

where λ is a constant. The constancy of λ is forced by (1) and in this case $E_B = \lambda S_{AB}$. Since, as we will see below, λ is real and in general $E_B \neq 0$ even when $S_{AB} = 0$, condition (12) corresponds to the minimum energy case [10]. To derive (12) we consider the quantity $E_B - \lambda S_{AB}$ which can be written as $\alpha\mathrm{d}((-)^n {}^*\mathrm{d}\alpha - \lambda\alpha)$ and varied with respect to α. (Actually, we may add to the right-hand side of (12) the n-form ${}^*\mathrm{d}\beta$ which contributes a total divergence to the field energy and may then be ignored.) If we define

$$I = \int \alpha \wedge {}^*\alpha$$

we have the Schwarz inequality $S_{AB}^2 \leqslant IE_B$. The equality holds precisely when (12) is satisfied. Since S_{AB} is a topological invariant (12), therefore, imposes the condition for minimal energy.

For the minimum energy case

$$\Delta\alpha = \lambda^2\alpha \tag{13}$$

where $\Delta = {}^*\mathrm{d}{}^*\mathrm{d} + \mathrm{d}^*\mathrm{d}^*$ is the Laplacian. If we consider α as a sum of $(n-1)$-forms then (13) is equivalent to $\Delta s_j = \lambda^2 s_i$ where the s_i are continuous functions. Takahashi studied this equation and he has shown that $\lambda^2 > 0$, and that the s_i may be viewed as coordinates of a sphere S^{2n-1} of radius $\sqrt{n-1}/|\lambda|$ [11]. Thus, $(\sum_i s_i^2)^{1/2}$ has a constant magnitude, i.e. the radius of the sphere.

The solution of (13) may also be presented in another way. Define f as an $(n-2)$-form which obeys the equations

$$\mathrm{d}^*f = 0 \qquad {}^*\mathrm{d}{}^*\mathrm{d}f = \lambda^2 f. \tag{14}$$

We introduce the l-form, $h = x^i\,\mathrm{d}x^i$ and form the following $(n-1)$-forms:

$$E = \mathrm{d}f \qquad F = {}^*\mathrm{d}(fh) \qquad G = {}^*\mathrm{d}F \qquad H = {}^*\mathrm{d}G. \tag{15}$$

By direct calculation each of these $(n-1)$-forms obeys (13). Hence α is a linear combination of these forms (the series (15) actually ends with G). By requiring that $\mathrm{d}\alpha = \lambda^*\alpha$ and $\mathrm{d}^*\alpha = 0$ we obtain the result

$$\alpha = \mathrm{constant}(F + \lambda^{-1}G) \tag{15}$$

which generalizes the result of Hansen [12].

The importance of the Hopf invariant can be seen as follows. For simplicity let $\mathrm{d}\alpha$ be a sum of a product of n l-forms

$$\omega_i\colon \mathrm{d}\alpha = f_i\omega^{i_1} \wedge \ldots \wedge \omega^{i_n}.$$

The set of characteristic vector fields ξ, defined by

$$i(\xi)\omega^{\mu} = 0 \qquad (\mu = i_1, \ldots, i_n)$$

forms a vector subspace of the tangent space of M over smooth functions. Here $i(\xi)\omega$ is the contraction of ω by ξ. If S is an $(n-1)$-surface on this subspace then $\int_S \alpha = 0$ and by Stokes' theorem $\int_{\partial^{-1}S} \mathrm{d}\alpha = 0$ where $\partial^{-1}S$ is the surface with S as boundary. However, the fact that $\int_S \alpha$ vanishes is not sufficient to guarantee that $\int_{\partial^{-1}S} \mathrm{d}\alpha$ also vanishes. The sufficient and necessary condition is, in fact, that $\alpha\,\mathrm{d}\alpha = 0$. Hence, unless $\alpha\,\mathrm{d}\alpha = 0$, it is not possible to construct homologically trivial surfaces in the space of characteristic vector fields.

Three observations can be made. First, since $\mathrm{d}\alpha = \lambda^*\alpha$ implies that $\alpha\,\mathrm{d}\alpha \neq 0$, it follows that the minimal energy configurations involves a non-trivial manifold structure. Second, when $n = 2$ the above considerations give the familiar Frobenius theorem for integrability. It is not clear whether this holds for higher n values. Third, equation (13) often occurs in the theory of harmonic and minimal maps and is, in fact, used as a working definition in that context [13]. This leads us to suspect that force-free fields must be related to minimal surfaces. We will see below that this is the case at least for $n = 2$.

The simplest case is $n = 2$, i.e. three-dimensional Euclidean space. The ϕ_i are zero forms which we write as $\phi_j = X_j^\mu W_\mu$ where μ are space indices and W_μ continuous functions. Equation (7) demands that $\partial_\mu W_\nu - \partial_\nu W_\mu = 0$, i.e. $W_\mu = \partial_\mu g$. Thus $\phi_j = i(X_j)\,\mathrm{d}g$ and

$$\alpha = \mathrm{d}g \tag{16}$$

where g is a continuous function. By (13) α satisfies $\Delta\alpha = \lambda^2\alpha$ and, according to Takahashi's theorem, $|\alpha|$ has constant magnitude. Now the gradient ∇g is metrically equivalent to $\mathrm{d}g$ and we may interpret

$$\hat{n} = \frac{\nabla g}{|\nabla g|} \propto \nabla g$$

as the normal field of the level surfaces of g. Since by (1) $\operatorname{div}\hat{n} = 0$, these surface are minimal surfaces (i.e. their mean curvature vanishes) [14].

A specific example is the magnetic field $\boldsymbol{B} = \sin kz\hat{\imath} + \cos kz\hat{\jmath}$. In terms of forms this translates to $\alpha = \frac{1}{k}(\sin kz\,\mathrm{d}x + \cos kz\,\mathrm{d}y)$ and we verify that $\mathrm{d}\alpha = k^*\alpha$, $\mathrm{d}^*\alpha = 0$ and $\Delta\alpha = k^2\alpha$. (α is not the vector potential of $\boldsymbol{B}$; it is $\boldsymbol{B}$ in the language of forms.) The corresponding function g is $g = \tan^{-1} x/y$ where $x^2 + y^2 = 1$ and the minimal surface is the helicoid. Another example with $n = 2$, is provided by the magnetic monopole in S^3, for which $\alpha = \mathrm{d}\chi + \cos\theta\,\mathrm{d}\phi$. It is well known that this corresponds to the principal bundle $S^3 \to S^2$ and that on S^2, $\mathrm{d}\alpha = {}^*\alpha$. S^2 is the minimal surface (imbedded in S^3) and the corresponding field has minimum energy with $S_{AB} = 1$ [9].

We have given these two examples, one in $\mathbb{R}^3$ and the other in S^3, because it is known that there cannot exist a compact minimal submanifold in Euclidean space [15]. However, when the enveloping manifold is compact a minimal immersion can exist.

Since force-free fields describe minimal surfaces, it follows that if ξ is any function in three-dimensional space and $\boldsymbol{B}$ the force-free magnetic field, then $0 = \xi\boldsymbol{B}\cdot\mathrm{d}\boldsymbol{a}$ where $\mathrm{d}\boldsymbol{a}$ is an element of area of the minimal surface S. Thus, by the divergence theorem

$$0 = \int_S \xi\boldsymbol{B}\cdot\mathrm{d}\boldsymbol{a} = \int_V \nabla\cdot\xi\boldsymbol{B}\,\mathrm{d}V = \int_V \nabla\xi\cdot\boldsymbol{B}\,\mathrm{d}V \tag{17}$$

where V is the volume enclosed by S. In this context the Hopf invariant is just the helicity integral

$$H = \int_V \boldsymbol{A}\cdot\boldsymbol{B}\,\mathrm{d}V$$

so that a gauge transformation $A \to A + \nabla\xi$ induces the change

$$H \to \int_V (A + \nabla\xi) \cdot B \, dV = \int_V A \cdot B \, dV. \tag{18}$$

That is, the helicity integral is also gauge invariant for force-free fields.

Finally, let us show how a constant of motion can be obtained. We revert back to general manifolds. Suppose v is a Killing field in a Euclidean space M. Let E_A be a set of orthonormal unit (frame) fields assigned to a minimal submanifold immersed in M. Consider the equation

$$\sum_A \bar{D}_{E_A} \langle v, E_A \rangle = \sum_A \langle \bar{D}_{E_A} v, E_A \rangle + \sum_A \langle v, \bar{D}_{E_A} E_A \rangle \tag{19}$$

where $\bar{D}_{E_A}$ is the connection defined on the submanifold. The last term on the right-hand side vanishes since, if D_{E_A} represents the connection in M and Π the second fundamental form, then

$$\begin{aligned} \sum_A \bar{D}_{E_A} E_A &= \sum_A (D_{E_A} E_A - \Pi(E_A, E_A)) \\ &= \sum_A D_{E_A} E_A \\ &= 0. \end{aligned} \tag{20}$$

The second line in (20) arises due to the fact that $\sum_A \Pi(E_A, E_A) = 0$ for minimal surfaces, and the third line arises from M being Euclidean. The first term on the right-hand side of (19) also vanishes because

$$\sum_A \langle \bar{D}_{E_A} v, E_A \rangle = \sum_A \langle D_{E_A} v, E_A \rangle - \sum_A \langle \Pi(E_A v), E_A \rangle. \tag{21}$$

The second term on the right-hand side of (21) is identically zero since Π is normal to the submanifold, and the first term also vanishes because v is a Killing field.

Hence

$$J_A = \langle v, E_A \rangle \tag{22}$$

satisfies

$$\bar{D}_A J_A = 0 \tag{23}$$

so J has constant value anywhere on the minimal surface. This result generalizes a similar result which is well known for geodesics and which we state: if v is a Killing field and γ a geodesic then $\langle v, \gamma' \rangle$ is constant along γ [16].

It is easy to check for the helicoid why (23) holds since the relevant Killing field are translations along and rotations about the helicoid axis: constancy of J is implied by symmetry.

In a recent work, Berger derived lower bounds on the energy of magnetic fields based on the twisting number of space curves [6]. This work centres on fields which have a helicoidal configuration. It is known that a helicoid can be obtained from a catenoid (also a minimal surface) by an appropriate bending process. We conjecture that Berger's bounds apply equally well to force-free fields of the catenoid type. This conjecture and a generalization of the connection between force-free fields and minimal surfaces point to work for the future.

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